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Classical Projections of Quantum Mechanics and the Limit $\hbar \longrightarrow 0$

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The convergence of the dynamics of classical projection to the dynamics of the classical limit is investigated for $\hbar \to 0$. A mistake from a previous paper is pointed out, and the correct version of the result is given. A new, similar result is presented if the function generating the Hamiltonian of both the classical projection and the classical limit is a polynomial.

1. INTRODUCTION

The Planck constant is a universal constant of nature. It is very small, and from the "classical" point of view, it can be considered to be very close to zero. An interesting mathematical tool for understanding the formal connection between quantum mechanics (QM) and classical mechanics (CM) is the limit $\hbar \to 0$. In some sense, CM should be a limit of QM for $\hbar \to 0$. Although this limit transition has no rigorous physical meaning, as a mathematical tool it gives an interesting formal relation between QM and CM.

Among the many papers concerning this topic are the classical work of Hepp (1974), Yaffe (1982), Bóna (1983), and Werner (1995). More reference of relevant papers concerning this topic are given in Werner (1995).

One of the purposes of the present paper is to give a corrected version of some results of the paper (Polakovič, 1998) where a mistake occurred in the proof of Theorem 2. Fortunately, it was possible to correct this result by adding a new assumption to the theorem. In the present paper, I give the correct version of Theorems 2 and 3 of Polakovič (1998). A new result is presented here. The approach given here is to take the limit transition $\hbar \to 0$ in the context of classical projections. Classical projections of QM were investigated by Bóna (1986; see also Bóna, 2000). In the present approach, the Hamiltonians of classical projections

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converge to the Hamiltonian of the classical limit for $\hbar \to 0$. The convergence of dynamics in this context is also investigated. This approach is similar to that proposed by Bóna (1983).

In Section 2, some preliminary considerations are presented. In Section 3, the correct version of the results of Polakovič (1998) is given. The results concern the convergence of dynamics for $\hbar \to 0$. More precisely, the dynamics of the classical projection converges to the dynamics of the classical limit for $\hbar \to 0$, and the Hamiltonian of the classical limit is from a specified class of bounded functions. The convergence is uniform on compact intervals in time. In Section 4, a similar new result is proved. Again the limit of dynamics of classical projections is investigated for $\hbar \to 0$. Here the function determining the Hamiltonian of the classical limit is a polynomial. The convergence of dynamics proved here is weaker than uniform, but stronger than pointwise.

2. PRELIMINARIES

Let U be a unitary irreducible representation of the Wey1–Heisenberg group in a separable Hilbert space $\mathcal{H}$. The generators $X_0, X_1, \ldots, X_{2n}$ can be chosen so that the canonical commutation relations (CCR) are satisfied:

$$X_0 = \hbar I, \qquad X_i = Q_i, \qquad X_{i+n} = P_i, \quad i = 1, \ldots, n.$$

Here Q_i and P_i are the coresponding operators of position and momentum in QM. The representation U can be considered as a projective representation of the additive group $\mathbb{R}^{2n}$ and

$$U_x = \exp\left(\frac{i}{\hbar} X \cdot S \cdot x\right)$$

where $X \cdot S \cdot x = X_j S_{jk} x_k$ and S is the standard symplectic $2n \times 2n$ matrix with elements S_{jk} given by

$$S_{jj+n} = -S_{j+nj} = 1, \quad j = 1, \ldots, n; \quad S_{jk} = 0 \quad \text{otherwise}$$

and

$$X = (X_1, \ldots, X_{2n}) = (Q, P) = (Q_1, \ldots, Q_n, P_1, \ldots, P_n),$$
$$x = (x_1, \ldots, x_{2n}) = (q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n).$$

We shall consider the Planck constant $\hbar$ to converge to 0. For this purpose, let us write $\lambda^2\hbar$ instead of $\hbar$ everywhere, where λ is a variable. Then the limit $\hbar \to 0$ will be represented by $\lambda \to 0$. For each value of $\lambda > 0$, we have

$$X^\lambda = \left(X_1^\lambda, \ldots, X_{2n}^\lambda\right) = \left(Q^\lambda, P^\lambda\right) = \left(Q_1^\lambda, \ldots, Q_n^\lambda, P_1^\lambda, \ldots, P_n^\lambda\right)$$

the set of operators of position and momentum, where we define

$$Q_i^\lambda = \lambda Q_i, \qquad P_i^\lambda = \lambda P_i, \quad i = 1, \ldots, n.$$

Then the operators Q_i^λ, P_i^λ form an irreducible representation of CCR if $\lambda^2\hbar$ is considered the value of the "Planck constant." The corresponding representation is

$$U_x^\lambda = \exp\left(\frac{i}{\lambda^2\hbar} X^\lambda \cdot S \cdot x\right) = \exp\left(\frac{i}{\hbar} X \cdot S \cdot \frac{x}{\lambda}\right) = U_{x/\lambda}.$$

Here U_x^λ are the Weyl operators for the given value of λ. Now let ψ be a convenient analytic vector of the representation U. Let us consider the orbit

$$O_\psi^\lambda = \left\{U_x^\lambda \psi; x \in \mathbb{R}^{2n}\right\}.$$

It is a symplectic manifold parametrized by a paramerer $x \in \mathbb{R}^{2n}$ and symplectomorphic to $\mathbb{R}^{2n}$ with the standard symplectic form (see Bona, 1986, 2000).

For each value of $\lambda > 0$, we can consider a classical Hamiltonian on O_ψ^λ:

$$h_\lambda(x) = \left(U_x^\lambda \psi, H^\lambda U_x^\lambda \psi\right)$$

where H^λ is a version of the formally given quantum Hamiltonian

$$H^\lambda = h\left(X_1^\lambda, \ldots, X_{2n}^\lambda\right).$$

Here h is a real function, which will play the role of the classical limit. The classical system with Hamiltonian $h_\lambda(x)$ will be called a classical projection (see Bona, 1986, 2000). In Polakovič (1998), the function h is denoted also by f. In the present paper, it will always be denoted by h, and the letter f will be reserved for other functions. In Polakovič (1998), h_λ was constructed according to a prescription proposed in Berezin and Shubin (1983). In Section 4 of the present paper, the function h will be a polynomial. In both cases, the following holds:

$$\lim_{\lambda \to 0} h_\lambda(x) = h(x).$$

This means that the Hamiltonian of the classical projection converges to the Hamiltonian of the classical limit for $\lambda \to 0$. There arises a natural question: Does the dynamics of the classical projection also converge to the dynamics of the classical limit? More precisely, let $x(0) = x_\lambda(0)$ be the same initial condition for the classical limit and the classical projection, with $\lambda > 0$ arbitrary. Let $x(t)$ be the evolved state of the classical limit at time t, and $x_\lambda(t)$ be the evolved state of the classical projection at time t. Does the time evolution $x_\lambda(t)$ converge to $x(t)$ in some sense for $\lambda \to 0$? In Section 3, the uniform convergence on compact intervals in time (for corresponding choice of the function h) is proved. In Section 4, the function h is a polynomial. A type of convergence that is perhaps weaker than uniform, but stronger than pointwise is proved in Section 4.

3. CORRECTION OF A PREVIOUS RESULT

In this section, we shall give the correct version of some results that were published in Polakovič (1998). In Polakovič (1998), h was taken to be a function

$$h(q, p) = \int e^{i(rq+sp)}\, \varphi(r, s)\, d^n r\, d^n s$$

where

$$r = (r_1, \ldots, r_n), \quad s = (s_1, \ldots, s_n)$$

and $\varphi(r, s)$ is the Fourier transform of h. As a result, some conditions on φ were found. The substitution of Q_i^λ, P_i^λ into the function h was given by

$$h(Q^\lambda, P^\lambda) = \int \exp[i(rQ^\lambda + sP^\lambda)]\, \varphi(r, s)\, d^n r\, d^n s$$

as was suggested in Berezin and Shubin (1983). In Polakovič (1998, Theorem 1) the limit transition

$$\lim_{\lambda \to 0} h_\lambda(x) = h(x)$$

was proved under some condition given on the function φ. The problem of convergence of the dynamics was also treated in Polakovič (1998). Unfortunately, there is a mistake in the proof of Theorem 2 of Polakovič (1998), but it is possible to correct this result by adding one conditon to its assumptions. We obtain the following.

Theorem 1. *Let*

$$\frac{\partial h_\lambda}{\partial x_i} \xrightarrow[\lambda \to 0]{} \frac{\partial h}{\partial x_i} \quad (i = 1, \ldots, 2n)$$

uniformly on $\mathbb{R}^{2n}$. *Let there exist positive constants* L_i, $i = 1, \ldots, 2n$, *such that*

$$\left| \frac{\partial h}{\partial x_i}(x) - \frac{\partial h}{\partial x_i}(y) \right| \le L_i |x - y|$$

for all $x, y \in \mathbb{R}^{2n}$. *Then the time evolution* $x_\lambda(t)$ *of systems with Hamiltonian* $h_\lambda(x)$ *will uniformly converge for* $\lambda \to 0$ *to the time evolution* $x(t)$ *of the system with the Hamiltonian of the classical limit* $h(x)$ *on the compact interval* $\langle 0, t_o \rangle$, *where* t_0 *is an arbitrary finite positive time such that all the evolutions exist for* $t \in \langle 0, t_0 \rangle$ *and the initial conditions are* $x_\lambda(0) = x(0)$ *for all* λ *that are sufficiently small.*

We shall use the following well known result.

Lemma 1 *(Gronwall's inequality), see Wiggins, 1990.* □ *Suppose the functions* $u(s)$ *and* $v(s)$ *are continuous and non-negative on the interval* $\langle t_0, t \rangle$, *and the* *function $c(s)$ is C^1 and non-negative on the interval $\langle t_0, t\rangle$ with*

$$v(t) \le c(t) + \int_{t_0}^{t} u(s)v(s)\,ds;$$

Then

$$v(t) \le c(t_0)\,\exp\left(\int_{t_0}^{t} u(s)\,ds\right) = \int_{t_0}^{t} \dot{c}(s)\left(\exp\int_{s}^{t} u(\tau)\,d\tau\right)\,ds.$$

Proof of Theorem 1. Let $f,\ f_\lambda\ :\ \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be given by

$$f(x) = \left(\frac{\partial h}{\partial p_1}(x), \ldots, \frac{\partial h}{\partial p_n}(x),\ -\frac{\partial h}{\partial q_1}(x), \ldots, -\frac{\partial h}{\partial q_n}(x)\right)$$

$$f_\lambda(x) = \left(\frac{\partial h_\lambda}{\partial p_1}(x), \ldots, \frac{\partial h_\lambda}{\partial p_n}(x),\ -\frac{\partial h_\lambda}{\partial q_1}(x), \ldots, -\frac{\partial h_\lambda}{\partial q_n}(x)\right)$$

so that the Hamiltonian equations for solutions $x(t)$, $x_\lambda(t)$ can be written

$$\dot{x}(t) = f(x(t)), \qquad \dot{x}_\lambda(t) = f_\lambda(x_\lambda(t)).$$

We obtain

$$\begin{aligned}\dot{x}_\lambda(t) - \dot{x}(t) &= f_\lambda(x_\lambda(t)) - f(x(t))\\ &= f_\lambda(x_\lambda(t)) - f(x_\lambda(t)) + f(x_\lambda(t)) - f(x(t)).\end{aligned}$$

By integrating from 0 to t_0 and considering absolute values, we obtain the following estimate:

$$\begin{aligned}|x_\lambda(t_0) - x(t_0)| \le &\int_0^{t_0} |f_\lambda(x_\lambda(t)) - f(x_\lambda(t))|\,dt\\ &+ \int_0^{t_0} |f(x_\lambda(t)) - f(x(t))|\,dt.\end{aligned}$$

Let $\varepsilon > 0$ be arbitarily small. From the uniform convergence

$$\frac{\partial h_\lambda}{\partial x_i} \to \frac{\partial h}{\partial x_i}$$

we see that there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ we have

$$|f_\lambda(x) - f(x)| < \varepsilon$$

The conditions given for constants L_i imply the existence of $L \ge 0$ such that

$$|f(x) - f(y)| \le L|x - y|$$

for all $x, y \in \mathbb{R}^{2n}$. Substituting these facts into the last estimate, we obtain

$$|x_\lambda(t_0) - x(t_0)| \leq \varepsilon(t_0 - 0) + L \int_0^{t_0} |x_\lambda(t) - x(t)|\, dt.$$

Now we can apply the Gronwall inequality for $c(s) = \varepsilon(s - 0), u(s) = L$, $v(s) = |x_\lambda(s) - x(s)|$, and taking the corresponding interval of integration, we obtain

$$\begin{aligned} |x_\lambda(t_0) - x(t_0)| &\leq \int_0^{t_0} \varepsilon \exp\left(\int_s^{t_0} L\, d\tau\right) ds \\ &= \frac{\varepsilon}{L}(e^{Lt_0} - 1) \leq \frac{\varepsilon}{L} e^{Lt_0}. \end{aligned}$$

As ε was arbitrarily small, the proof is complete. □

Now we can give a correct version of Theorem 3 in Polakovič (1998). It is again sufficient to add one condition on the function h. Let us denote (as in Polokovič, 1998)

$$y_i = r_j, \qquad y_{n+j} = s_j, \quad j = 1, \ldots, n.$$

We obtain the following result.

Theorem 2. *Let the Fourier transform φ of the function h satisfy $\varphi \in L^1(\mathbb{R}^{2n})$, $y_j\varphi \in L^1(\mathbb{R}^{2n})$, $j = 1, \ldots, 2n$. Let there exist positive constants L_j, $j = 1, \ldots, 2n$, such that*

$$\left| \frac{\partial h}{\partial x_j}(x) - \frac{\partial h}{\partial x_j}(y) \right| \leq L_j |x - y|$$

for all $x, y \in \mathbb{R}^{2n}$. Then the dynamics $x_\lambda(t)$ converges uniformly for $\lambda \to 0$ to $x(t)$ on the intervals $\langle 0, t_0 \rangle$ for the initial conditions $x_\lambda(0) = x(0)$. In particular, this condition is satisfied for arbitrary $h \in \mathcal{S}(\mathbb{R}^{2n})$ (Schwartz space).

Remark. The added condition makes the set of acceptable functions h smaller. In any event, the Schwartz space is contained in it.

4 POLYNOMIAL HAMILTONIANS

Let now h be a polynomial in variables $x_1, \ldots, x_{2n}$. So

$$H^\lambda = h\big(X_1^\lambda, \ldots, X_{2n}^\lambda\big)$$

is the quantum Hamiltonian for given $\lambda > 0$. Here we shall not discuss the condition of symmetry of H^λ; it would be satisfied by choosing the corresponding ordering of the operators $X_1^\lambda, \ldots, X_{2n}^\lambda$ in this expression.

Let us now compute the expression for $h_\lambda(x) = h_\lambda(x_1, \ldots, x_{2n})$. We shall use the identity

$$\left(U_x^\lambda\right)^{-1} X_i^\lambda U_x^\lambda = X_i^\lambda + x_i I.$$

(For proof, see Polakovič, to appear). Let us, for simplicity, consider h to be a monomial; for a polynomial we have analogous results. So let

$$h(x) = x_{i_1} x_{i_2} \ldots x_{i_k}$$
$$i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, 2n\}.$$

Now

$$\begin{aligned}
h_\lambda(x) &= \left(U_x^\lambda \psi, h\left(X_1^\lambda, \ldots, X_{2n}^\lambda\right) U_x^\lambda \psi\right) \\
&= \left(U_x^\lambda \psi, X_{i_1}^\lambda X_{i_2}^\lambda \ldots X_{i_k}^\lambda U_x^\lambda \psi\right) \\
&= \left(\psi, \left(U_x^\lambda\right)^{-1} X_{i_1}^\lambda X_{i_2}^\lambda \ldots X_{i_k}^\lambda U_x^\lambda \psi\right) \\
&= \left(\psi, \left(U_x^\lambda\right)^{-1} X_{i_1}^\lambda U_x^\lambda \left(U_x^\lambda\right)^{-1} X_{i_2} U_x^\lambda \ldots \left(U_x^\lambda\right)^{-1} X_{i_k}^\lambda U_x^\lambda \psi\right) \\
&= \left(\psi, \left(X_{i_1}^\lambda + x_{i_1} I\right)\left(X_{i_2}^\lambda + x_{i_2} I\right) \ldots, \left(X_{i_k}^\lambda + x_{i_k} I\right)\psi\right) \\
&= \left(\psi, (\lambda X_{i_1} + x_{i_1} I)(\lambda X_{i_2} + x_{i_2} I) \ldots (\lambda X_{i_k} + x_{i_k} I)\psi\right).
\end{aligned}$$

A simple computation gives

$$h_\lambda(x) = x_{i_1} x_{i_2} \ldots x_{i_k} + A_1 \lambda P_1(x) + A_2 \lambda^2 P_2(x) + \cdots + A_k \lambda^k P_k(x)$$

where $P_i(x)$ are polynomials in $x_1, \ldots, x_{2n}$ of the $(k - i)$th degree and A_i, $i = 1, \ldots, k$, are real constants. We see immediately that if h is an arbitrary polynomial of degree k, then

$$h_\lambda(x) = h(x) + A_1 \lambda P_1(x) + A_2 \lambda^2 P_2(x) + \cdots + A_k \lambda^k P_k(x)$$

where A_i, $P_i(x)$ are as above. From this expression, it can immediately be seen that

$$\lim_{\lambda \to 0} h_\lambda(x) = h(x)$$

(see also Bona, 1986), so the Hamiltonian of the classical projection converges to the Hamiltonian of the classical limit for $\lambda \to 0$.

Now, if we form the Hamilton equations for the Hamiltonian $h(x)$, we obtain

$$\dot{x} = g(x) \tag{1}$$

where

$$g(x) = (g^1(x), g^2(x), \ldots, g^{2n}(x))$$

and $g^i(x)$ is a polynomial in $x_1, \ldots, x_{2n}$ [a partial derivative of $h(x)$]. For Hamiltonian $h_\lambda(x)$, we obtain the Hamilton equation in the form

$$\dot{x} = g(x) + \lambda g_1(x) + \lambda^2 g_2(x) + \cdots + \lambda^k g_k(x) \tag{2}$$

where

$$g_j(x) = \left(g_j^1(x), \ldots, g_j^{2n}(x)\right)$$

and $g_j^i(x)$ is a polynomial in variables $x_1, \ldots, x_{2n}$ (it may be equal to 0).

Let now $x(t)$ be the solution of (1) and $x_\lambda(t)$ the solution of (2), where $x(0) = x_\lambda(0)$ for all relevant values of λ. We shall prove that $x_\lambda(t)$ converges to $x(t)$ in some sense for $\lambda \to 0$. For the proof, we shall need some basic facts about ordinary differential equations. They will be taken from Greguš *et al.* (1985). As this book is not available in an English language version, we shall present here the basic propositions from it as lemmas.

Let us take an ordinary differential equation

$$\dot{y} = f(t, y) \tag{3}$$

with initial condition

$$y(t_0) = y_0 \tag{4}$$

where $y \in \mathbb{R}^p$, $t \in \mathbb{R}$, and $(t, y) \in O \subseteq \mathbb{R}^{p+1}$, where O is an open set.

Definition 1 (see Greguš *et al.*, 1985). We say that the function $f : 0 \to \mathbb{R}^p$ satisfies the local Carathéodory conditions in O if, for each point $(t_0, y_0) \in O$ there exists a set

$$V(t_0, y_0; a, b) = \{(t, y) \in \mathbb{R}^{p+1}; \quad |t - t_0| \leq a, |y - y_0| \leq b\} \subseteq O$$

where $a > 0, b > 0$ such that for each $(t_1, y_1) \in V(t_0, y_0; a, b)$:

(a) The function $f(t, y_1)$ is measurable on $\langle t_0 - a, t_0 + a\rangle$ as a function of t,
(b) The function $f(t_1, y)$ is continuous in $\{y \in \mathbb{R}^p; |y - y_0| \leq b\}$ as a function of y,
(c) There exists a non-negative function $m \in L(\langle t_0 - a, t_0 + a\rangle)$ (Lebesgue integrable) such that $|f(t, y)| \leq m(t)$ for all $(t, y) \in V(t_0, y_0; a, b)$.

Definition 2 (see Greguš *et al.*, 1985). The solution $y : I \to \mathbb{R}^p$ of Eq. (3) is said to be complete if and only if for each solution $z : J \to \mathbb{R}^p$ of (3) and $I \subseteq J$, $z(t) = y(t)$ on I, has $z = y$ (so $J = I$). The interval I is called the maximal interval of the existence of the solution.

Definition 3 (see Greguš *et al.*, 1985). Let $O \subseteq \mathbb{R}^{p+1}$ be open, $(t_0, y_0) \in O$. Equation (3) is said to have the property of uniqueness in (t_0, y_0) if the following holds: if y_1, y_2 are solutions of (3) on the interval J such that $y_1(t_0) = y_2(t_0) = y_0$, then $y_1 = y_2$ on J.

Lemma 2 *(see Greguš et al., 1985, Theorem 5.2). If the set O is open, then the maximal interval of the existence of the solution is always open.*

Lemma 3 *(see Greguš et al., 1985, Theorem 5.3). If the differential Eq. (3) has the property of uniqueness in each point of the set O, then there exists exactly one complete solution of (3), (4).*

Let $O \subseteq \mathbb{R}^{p+1}$ be a nonempty open set, the function $f : O \to \mathbb{R}^p$ satisfy the local Carathéodory conditions in O and Eq. (3) have the property of uniqueness in each point of O. According to Lemmas 2 and 3, for each $(t_0, y_0) \in O$, there exists exactly one complete solution of (3) with initial condition (4) that is defined on an open interval. This solution will be denoted by $y(t; t_0, y_0)$ and its interval of definition by $(a(t_0, y_0), b(t_0, y_0))$. The following holds:

Lemma 4 *(see Greguš et al., 1985, Lemma 6.1). Let $O \subseteq \mathbb{R}^{p+1}$ be a nonempty open set, the function $f : O \to \mathbb{R}^p$ satisfy the local Carathéodory conditions in O, and Eq. (3) have the property of uniqueness in each point of O.*

Let the sequence of points $(t_k, y_k) \in O$ satisfy $\lim_{k\to\infty}(t_k, y_k) = (t_0, y_0) \in O$. Then for each compact interval $\langle c, d\rangle \subset (a(t_0, y_0), b(t_0, y_0))$ there exists k_0 such that for each $k \geq k_0$, one has $a(t_k, y_k) < c < d < b(t_k, y_k)$ and the sequence $y(t; t_k, y_k)$ uniformly converges to $y(t; t_0, y_0)$ on $\langle c, d\rangle$ for $k \to \infty$.

Definition 4 (see Greguš *et al.*, 1985, Definition 2.8). Let $f : D \to \mathbb{R}^n$, $D \subseteq \mathbb{R}^{n+1}$. We say that the function $f = f(t, y)$ is Lipschitz in D if there exists a constant $L > 0$ such that for two arbitrary points (t, y) and $(t, \overline{y})$ in D the following holds:

$$|f(t, y) - f(t, y)| \leq L|y - \overline{y}|.$$

Lemma 5 *(see Greguš et al., 1985, Theorem 2.17). Let f be continuous and Lipschitz on $O \subseteq \mathbb{R}^{n+1}$, O open set, $(t_0, y_0) \in O$. Then Eq. (3) has the property of uniqueness in (t_0, y_0).*

Let us now apply the preceding basic results about differential equations to Eqs. (1) and (2). We can consider them as a single equation because (1) is in fact

(2) where we set $\lambda = 0$. Equation (2) can be written

$$\dot{x} = G(\lambda, x)$$

where the components of $G(\lambda, x)$ are polynomials in $x_1, \ldots, x_{2n}, \lambda$. If we add the equation $\dot{\lambda} = 0$, we obtain the system of equations

$$\dot{x} = G(\lambda, x), \qquad \dot{\lambda} = 0$$

with initial conditions

$$x(t_0) = x_0, \qquad \lambda(t_0) = \lambda_0,$$

which is equivalent to the system (2) if $\lambda = \lambda_0$ and $x(t_0) = x_0$ are chosen. Let $\lambda \in (-1, 1)$. If we denote

$$y = (x, \lambda) = (x_1, \ldots, x_{2n}, \lambda),$$
$$y_0 = (x_0, \lambda_0),$$

we can write Eq. (2) (if $\lambda = \lambda_0$) as

$$\dot{y} = f(t, y) = f(y), \qquad y(t_0) = y_0$$

where the components of $f(y)$ are polynomials in $x_1, \ldots, x_{2n}, \lambda$.

Let now $t_0 = 0$ and consider the solution $x(t)$ of (1) with initial condition $x(0) = x_0$. Let the solution exist for $t \in \langle 0, T + \varepsilon \rangle$, where $T > 0$, $\varepsilon > 0$. Let

$$M = \max_{t \in \langle 0, T+\varepsilon \rangle} |x(t)|,$$
$$K = (-2M, 2M)^{2n},$$
$$I = (-T - \varepsilon, T + \varepsilon).$$

Let us define the open set

$$O = I \times K \times (-1, 1)$$

[which means $t \in I$, $x \in K$, $\lambda \in (-1, 1)$]. The function f will be defined on O, $f : \overline{O} \to \mathbb{R}^{2n+1}$. Clearly, $\overline{O}$ (the closure of O) is compact. The function f is continuous in $\overline{O}$, so the local Carathéodory conditions for f are satisfied in O. The equation

$$\dot{y} = f(t, y)$$

has the property of uniqueness in O because it is continuous and Lipschitz in O according to Lemma 5. The Lipschitz condition is satisfied because the components of f are polynomials and $\overline{O}$ is compact.

So the conditions of Lemma 4 for the equation $\dot{y} = f(t, y)$ are satisfied. Let us put $t_0 = 0$, $y_0 = (x_0, 0)$ (so $\lambda_0 = 0$), $t_k = 0$, $y_k = (x_0, \lambda_k)$, where $\lambda_k \xrightarrow[k\to\infty]{} 0$. One has $b(t_0, y_0) > T + \varepsilon$, $a(t_0, y_0) < 0$ according to Lemma 2. We can put $c = 0$, $d = T$

and from the statement of Lemma 4, we immediately obtain the main result of this section:

Theorem 3. *Let $x(t)$ be the solution of (1), $x(0) = x_0, t \in \langle 0, T + \varepsilon\rangle$, where $T > 0, \varepsilon > 0$. Let $x_\lambda(t)$ be the solution of (2) for a given value of λ, $x_\lambda(0) = x_0$. Let $\lambda_k \to 0$ if $k \to \infty$. Then there exists k_0 such that for each $k > k_0$, the solution $x_{\lambda_k}(t)$ is defined on $\langle 0, T\rangle$ and the sequence $x_{\lambda_k}(t)$ uniformly converges to $x(t)$ on $\langle 0, T\rangle$ for $k \to \infty$.*

Remark. This is actually the main result because Eq. (1) is the Hamilton equation for the classical limit and Eq. (2) is the Hamilton equation of the classical projection if there is given a polynomial Hamiltonian. So the time evolution of the classical projection was shown to converge to the time evolution of the classical limit in some sense for $\lambda \to 0$. The shown convergence is weaker than the uniform convergence on compact intervals in time, but it is clearly stronger than the simple pointwise convergence. The question remians open of whether one can also prove the uniform convergence on compact intervals in time.

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